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Quantum stabilization of Z-strings, a status report on $D = 3 + 1$ dimensions

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Abstract
We investigate an extension to the phase shift formalism for calculating one-loop determinants. This extension is motivated by requirements of the computation of Z-string quantum energies in $D = 3+1$ dimensions. A subtlety that seems to imply that the vacuum polarization diagram in this formalism is (erroneously) finite is thoroughly investigated.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction and motivation

Z-strings were first discovered as solutions to the classical field equations of the electroweak standard model by Nambu [1] in the context of bound pairs of magnetic monopoles. Later on they were rediscovered—as independent objects in their own right—by Vachaspati [2].

The main point of interest in our study of Z-strings is their stability. If they are stable, they might be relevant for a variety of reasons: they would be the only static solitons in the standard model, networks of Z-strings might play an important role in baryogenesis [3], and they would be a possible source of the primordial magnetic field. For a general overview of applications and properties of Z-strings along with a large collection of references cf [4].

Because there are no configurations with nontrivial topology in the electroweak model, stability of Z-strings is only possible on energetic grounds. Classically, Z-strings are unstable for the physical values of the parameters of the electroweak model. Since fermions bind

[5] Talk presented at QFEXT'07 by OS.

strongly to the core of the Z-string, it may be possible to achieve stability by occupying N fermionic bound states that are generated in the background of the Z-string so that the resulting total energy of the system is less than the energy of N free fermions. A consistent $\hbar$ expansion then requires the inclusion of the contribution of the fermionic determinant to the energy.

In $D = 3 + 1$ dimensions, renormalization issues make the investigation of the Z-string difficult since it is non-perturbative. Investigations have either failed to draw convincing conclusions [5] or focused on $D = 2 + 1$ dimensions [6, 7]. Here we will discuss attempts to solve the problems posed by these earlier investigations.

The structure of this paper is as follows: in section 2, we discuss our method for computing the fermion determinant and the problems posed by the Z-string for our approach. We discuss the possible solutions in $D = 2 + 1$ dimensions and the extension necessary for $D = 3 + 1$ dimensions. In section 3, we investigate whether the extension proposed for the Z-string works in the case of a simple boson model. Problems involving the second-order Born approximations are discussed and resolved by comparing different formulations of our approach. In section 4 we present some conclusions and an outlook.

2. Technical prerequisites

The Z-string has the following structure:

$$\phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} = v \begin{pmatrix} 0 \\ f_H(\rho)\,\mathrm{e}^{\mathrm{i}\varphi} \end{pmatrix}, \qquad g\vec{Z} = \frac{\hat{\varphi}}{\rho} 2 f_G(\rho), \tag{1}$$

with ϕ being the iso-spinor Higgs field and $\vec{Z}$ the only non-vanishing component of the $SU(2) \times U(1)$ gauge fields. As usual, ρ and φ denote the radial and angular coordinate in the plane perpendicular to the Z-string and v is the vacuum expectation value of the Higgs field. The string configuration is defined in two spatial dimensions. It is translationally invariant in any additional space coordinate, which we call *flat* dimension(s). Finite classical energy (for $D = 2 + 1$, or classical energy per unit length in $D = 3 + 1$) requires both $f_H \to 0$ and $f_G \to 0$ for $\rho \to 0$. Furthermore, the kinetic energy of the Higgs must fall off sufficiently fast at large distances,

$$|(\partial_\mu - \mathrm{i}gZ_\mu)\phi|^2 = \mathcal{O}(1/\rho^{2+\epsilon}), \qquad \epsilon > 0 \qquad \text{for} \quad \rho \to \infty. \tag{2}$$

In particular, this implies that $\phi^\dagger\phi = |f_H|^2 \to 1$ and $f_G \to 1$ at $\rho \to \infty$. The condition (2) clearly mixes different orders in perturbation theory (PT) and already indicates that if one sticks to a *fixed* order in PT one will end up with IR divergent quantities. In the systematic expansion of arbitrarily many fermion species and at next to leading order in $\hbar$, the quantum correction to the energy, i.e. the vacuum polarization energy, may be computed from the fermion determinant in the background of the potential $V(\vec{x})$ that is generated by the string. In turn, this determinant is given in terms of scattering data from this potential [8]. This formulation is effective for renormalization because the nth order contribution in the Born expansion equals the Feynman diagram for the fermion loop with n insertions of V. To render the integral over scattering data finite, we subtract enough Born terms and add them back in as Feynman diagrams. The renormalization of the latter is adopted from perturbation theory and is standard. Hence the vacuum polarization energy is given by

$$E_{\text{vac}} = \frac{1}{2}\sum_{b.s.} f_1\left(\omega_j^{b.s.}\right) - \frac{1}{2\pi}\int \mathrm{d}k\, f_2(k) \sum_M \left[\delta_M(k) - \sum_{n=1}^N \delta_M^{(n)}(k)\right] + \sum_{n=1}^N E_{\text{FD}}^{(n)} + E_{\text{CT}}, \tag{3}$$

where $\omega_j^{b.s.}$ denotes the fermion bound state energies, δ_M the (full) phase shift in angular momentum channel M, $\delta_M^{(n)}$ its nth Born approximation, N denotes the number of Born approximations necessary to render the momentum integral finite, $E_{\mathrm{FD}}^{(n)}$ is the energy contribution computed from Feynman diagrams with n external legs and E_{CT} is the energy resulting from the counter terms. The functions $f_{1,2}$ vary with the number of flat dimensions [9]. For the Z-string in $D = 2 + 1$, the number of flat dimensions is 0 and we obtain $f_1(\omega) = m - |\omega|$, $f_2(k) = -k/\sqrt{k^2 + m^2}$. For $D = 3 + 1$, the number of flat dimensions is 1 and we obtain $f_1(\omega) = (\omega^2(1 + \ln(\omega^2/m^2)) - m^2)/(4\pi)$, $f_2(k) = (k/2\pi)\ln(1 + k^2/m^2)$. Of course, the Feynman diagram and counter term contribution also differ for $D = 2 + 1$ and $D = 3 + 1$, but these expressions are well known and will not be repeated here. When computing the Born approximation for scattering in the Z-string background, one encounters two problems: the full scattering problem is decomposed into distinct channels of finite size. The decomposition of the full scattering problem utilizes the generalized axial symmetry of the Hamiltonian h in a Z-string background

$$[h, J_3 - nT_3\gamma_5] = 0, \tag{4}$$

where J_3 is the fermionic angular momentum, T_3 is an isospin generator and n is the (integer) flux of the Z-string. Since γ_5 does not commute with the free Hamiltonian h_0, we have $[h_0, J_3 - nT_3\gamma_5] \neq 0$. Thus, in contrast to standard problems, the symmetries of the problem with background field are *not a subset* of the symmetries of the free problem, but instead are incompatible with them. Thus, the setup for perturbation theory (which considers the Hamiltonian $h_\epsilon = h_0 + \epsilon V$) has no axial symmetry *at all*. The second problem is related to the aforementioned IR problems of strings. Only special (gauge invariant) combinations of Feynman diagrams of different perturbative orders result in IR finite quantities. However, Born approximations correspond to sums of Feynman diagrams of definite order in PT and hence most likely will suffer from IR divergences.

The case of $D = 2+1$ is special. The only counter term that has a divergent co-efficient—$|\phi|^2$—is of a definite order in PT. In this case, we are able to use gauge invariance to argue that any background that has the same $|\phi|^2$ will suffer from the same divergence and hence the Born approximation for some other *fake* background can be used to subtract the large-momentum tails of the full phase shifts. So we can choose a background that has J_3 as a symmetry generator but no IR divergences [7].

In $D = 3+1$ this procedure does not work because we now have more counter terms with divergent coefficients that mix different orders of PT. Furthermore, since there are additional constraints on the profiles, it is not clear how to construct fake background solutions which have no winding and thus the same symmetry as the free Hamiltonian—if it is possible at all.

At this point it is worthwhile to remember *why* we want to subtract Born approximations in the first place: first, we have to subtract off the large-momentum behaviour of the summed phase shifts in order to ensure UV convergence of the momentum integral. Second, with Born approximations we know exactly what to add back in so that the overall value of the determinant does not change, namely the corresponding Feynman diagrams. Keeping these reasons in mind, we can extend our formalism substantially: instead of subtracting the Born approximations of the original theory with the original background fields, we can subtract Born approximations from an arbitrary theory as long as (a) we add back in the Feynman diagrams from the same theory and (b) that theory has the same divergences (e.g. in dimensional regularization) as the original theory. In the remainder of this paper, we test this proposition.

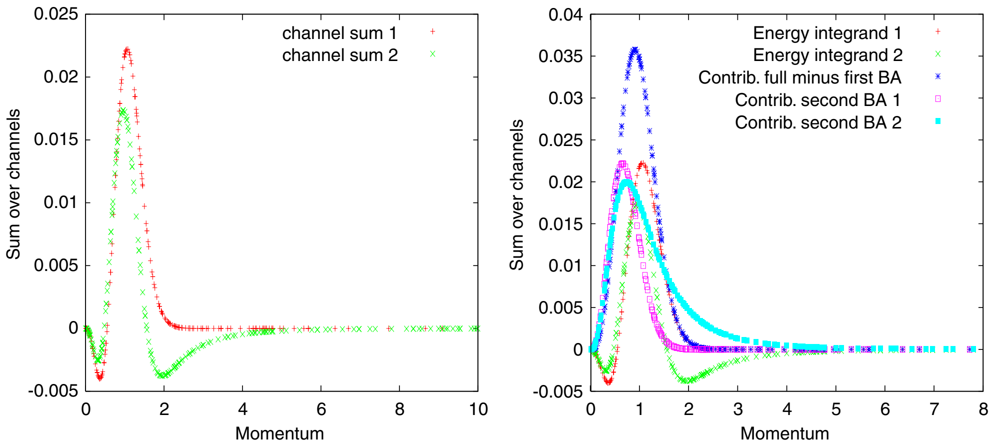

Figure 1. The left panel shows $k\ln(1+k^2/m^2)\sum_M \frac{1}{\pi}[\delta_M(k;\sigma_1)-\delta_M^{(1)}(k;\sigma_1)-\delta_M^{(2)}(k;\sigma_i)]$ for $i = 1, 2$. The right panel shows the decomposition furthermore into $k\ln(1+k^2/m^2)\sum_M \frac{1}{\pi}[\delta_M(k;\sigma_1)-\delta_M^{(1)}(k;\sigma_1)]$ and $k\ln(1+k^2/m^2)\sum_M \frac{1}{\pi}\delta_M^{(2)}(k;\sigma_i)$.

3. Fake subtractions

We consider two bosonic theories in $D = 3 + 1$ that differ in the string-like background potential they are coupled to

$$\mathcal{L}_{1,2} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{2}\sigma_{1,2}\phi^2 \tag{5}$$

with $\sigma_1(\vec{x}) = \alpha_1 \exp\left(-\rho^2/w_1^2\right)$ and $\sigma_2(\vec{x}) = \alpha_2\exp(-\rho/w_2)$. Demanding identical UV divergences requires

$$\int \mathrm{d}^3x\,\sigma_1(\vec{x}) = \int \mathrm{d}^3x\,\sigma_2(\vec{x}) \qquad \text{and} \qquad \int \mathrm{d}^3x\,\sigma_1^2(\vec{x}) = \int \mathrm{d}^3x\,\sigma_2^2(\vec{x}). \tag{6}$$

If our proposal of a fake Born subtraction works, the renormalized vacuum polarization energy in (3) should be the same, whether we use the original model 1 or its fake version 2 for the subtraction. Since the bound states are not affected by the subtraction, it is sufficient to consider the part

$$\tilde{E}_i = \frac{1}{2}\int \mathrm{d}k\,k\ln\left(1+\frac{k^2}{m^2}\right)\sum_M \frac{1}{\pi}\left[\delta_M - \sum_{n=1}^{2}\delta_M^{(n)}(k;\sigma_i)\right] + \sum_{n=1}^{2} E^{(n)}_{\mathrm{FD,ren}}(k;\sigma_i), \tag{7}$$

where the full phase shift is always computed in the original model 1, while the subtraction is made with model 1 Born approximations ($\tilde{E}_1$) or with the fake model 2 Born approximations ($\tilde{E}_2$), respectively. Our proposal amounts to claiming that $\tilde{E}_1 = \tilde{E}_2$.

It can be shown, using a Bessel function identity [8], that $\sum_{M=0}^{\infty}\delta_M^{(1)}(k;\sigma_i) \propto \langle\sigma_i\rangle k^{q-2}$ where q is the number of non-flat spatial dimensions. By numerical calculation we can furthermore show that

$$\frac{1}{2}\int \mathrm{d}k\,k\ln\left(1+\frac{k^2}{m^2}\right)\sum_M \frac{1}{\pi}\left[\delta_M^{(2)}(k;\sigma_2) - \delta_M^{(2)}(k;\sigma_1)\right] = E^{(2)}_{\mathrm{FD,ren}}(\sigma_2) - E^{(2)}_{\mathrm{FD,ren}}(\sigma_1). \tag{8}$$

This shows that as long as the renormalized energy is finite, both ways of subtracting Born approximations lead to the same result. In the left panel of figure 1 we see that the renormalized

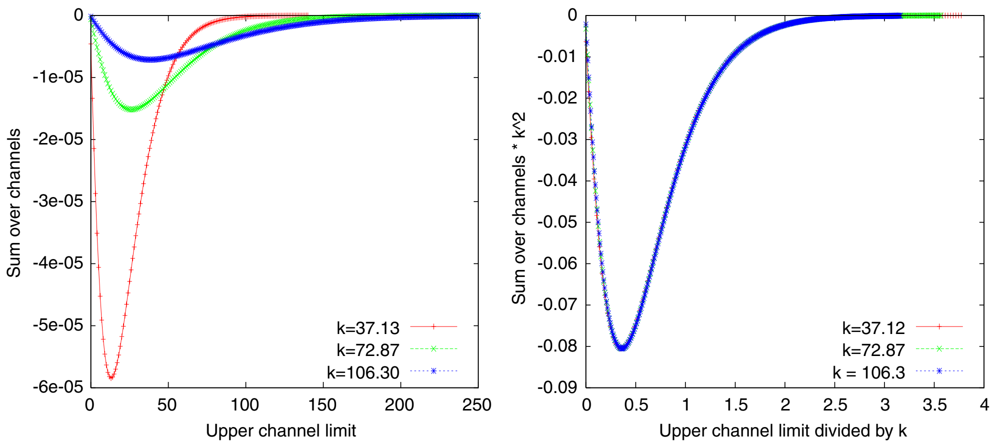

Figure 2. The left panel shows $S_N^{(2)}(k) = \sum_{M=0}^{N} \delta_M^{(2)}(k; \sigma_2)$ for different values of k, the right panel shows $k^2 S_N^{(2)}(k)$ plotted versus N/k.

energy is going to be finite, since there is no large-momentum tail that might impede the existence of the momentum integral. The right panel, however, shows something troubling: ordinarily, we would think that (7) with just the first term of the Born series subtracted still contains the log-divergent second-order Feynman diagram. The Feynman series then suggests the same large k behaviour for the integrand of (7) when the expression in square brackets is replaced by the second-order term of the Born series. From the right panel of figure 1, one can clearly see that this expectation is wrong: both contributions to the integrand in (7) fall off faster than $1/k^3$ individually. At first sight this indicates that the resulting integral would be finite [10, 11]. This is in strong contradiction to the fact the equivalent second-order Feynman diagram is indeed UV divergent. As we will demonstrate in the rest of this paper the catch comes from an incorrect treatment of the non-uniformly convergent sum and integral in (7).

First of all, it is worth noting that the second-order Born approximation phase shifts are not small by themselves, but they sum up to something small, cf figure 2. Secondly, it is worthwhile to recall the formulation of functional determinants in terms of scattering data [12, 13]. Strictly this is possible only in the upper half of the complex k plane and not on the real k-axis. Furthermore, that derivation also suggests that instead of integrating over a channel sum, one should integrate over momentum first to get a per-channel contribution to the energy and then sum over channels. The analytic properties of scattering data ensure that the contribution of a prescribed channel to the vacuum polarization energy can be identically computed as an integral over real or imaginary momentum. We have verified this result numerically. Of course, the corresponding integrands are not expected to be identical. Consequently, the momentum integrands obtained from summing over channels first are expected to be different as well. The momentum integrand of the second-order Born approximation phase shift[1] on the imaginary axis shows exactly the expected $1/t$ fall off[2] for a logarithmically divergent integral, cf left panel of figure 3. In the right panel, one can see that for all channels the second-order Born approximations on the imaginary axis have the same

[1] More precisely one should talk about the imaginary parts of the logarithm of the Jost function, because the concept of a phase shift cannot be extended to the complex k plane.

[2] The momentum on the imaginary axis is denoted t, i.e., $k = \mathrm{i}t$ with t real.

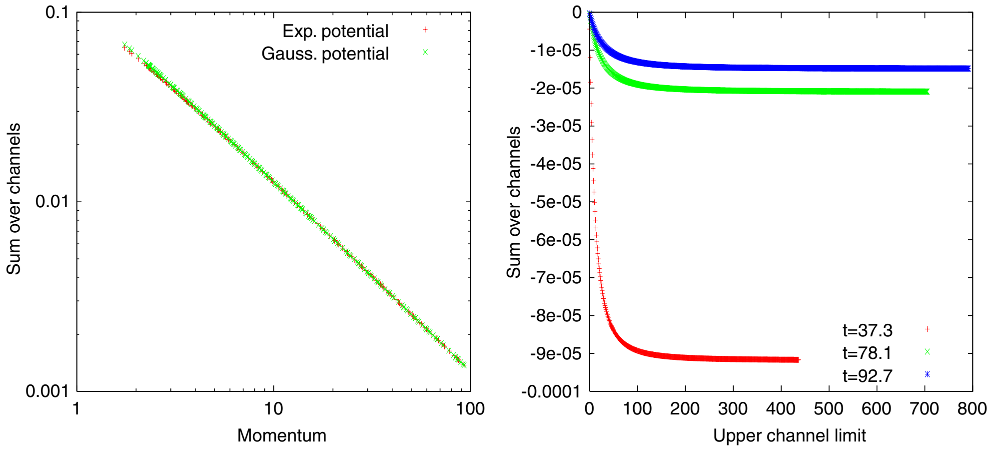

Figure 3. The left panel shows $t\sum_M \tilde{\delta}_M^{(2)}(t;\sigma_i)$ (which is the expression on the imaginary axis corresponding to $k\ln(1+k^2/m^2)\sum_M \delta_M^{(2)}$ on the real k-axis) for different values of t and $i=1,2$. The right panel shows $\sum_{M=0}^{N}\tilde{\delta}_M^{(2)}(t;\sigma_2)$ plotted versus N for different values of t.

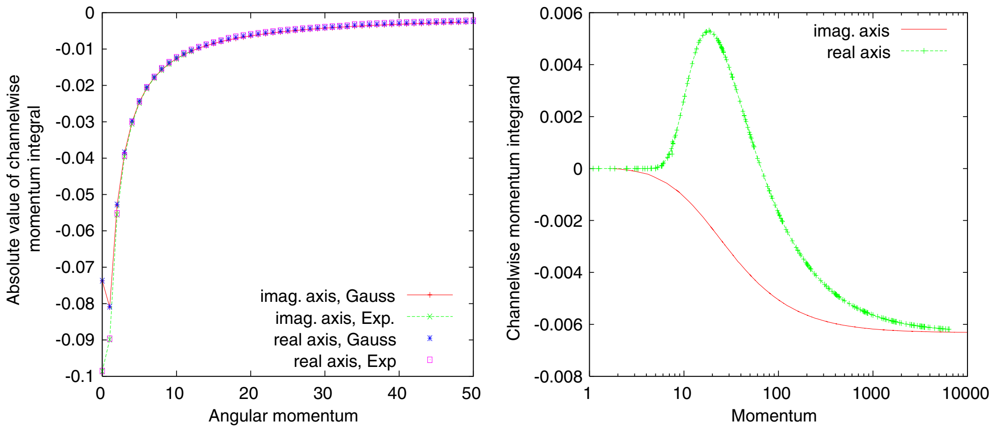

Figure 4. The left panel shows $\int \mathrm{d}k\, k\ln(1+k^2/m^2)\delta_M^{(2)}(k)$—denoted as 'real axis'—and $\int \mathrm{d}t\, t\tilde{\delta}_M^{(2)}(t)$—denoted as 'imaginary axis'—for both σ_1 and σ_2 as potentials. Note that for larger values of M, the curves for $\sigma_{1,2}$ coincide and show the expected $1/M$ fall off. The right panel shows the momentum integrand for both real and imaginary axis for channel $M=10$.

sign and no cancellations are present. Moreover, our general experience together with our numerical investigations in this particular case suggest that the sum over channels is uniformly convergent and we can freely interchange the order of channel summation and momentum integral.

The left panel of figure 4 shows clearly that the problem on the real axis is the interchange of integration and summation, because by integrating first over momentum and then summing over channels we obtain the expected logarithmic divergence, whereas in the other order the result is apparently finite. The right panel shows how the (identical) values on the real and imaginary axis are obtained—on the imaginary axis we see a monotonic approach to the final value, while on the real axis we see a zero crossing. This zero is not accidental, but rather a

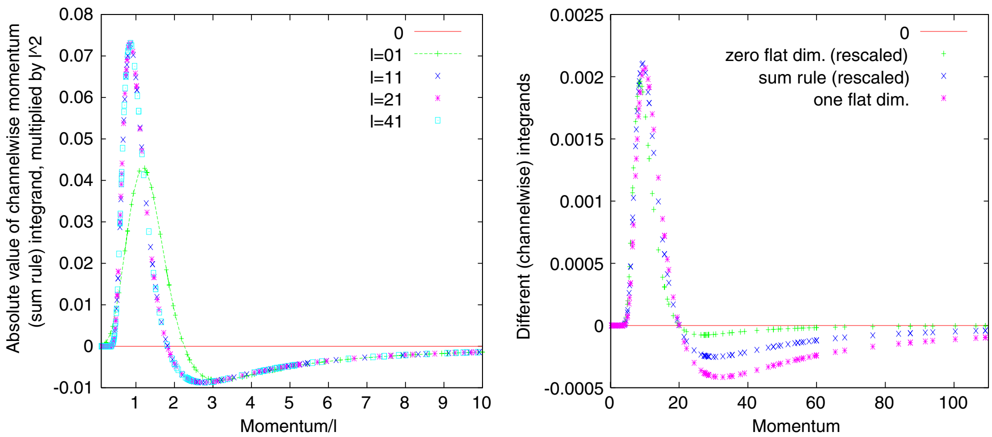

Figure 5. The left panel shows another scaling rule; the most important point is that the zero crossing moves outwards on the real k-axis as M increases. The right panel compares the integrands of the per-channel contribution to the energy with zero and one flat dimension to the integrand of the sum rule.

consequence of the fundamental sum rule

$$\int \mathrm{d}k\, k\delta_l^{(2)}(k) = 0, \tag{9}$$

proved in [14]. On the imaginary axis no corresponding sum rule exists. The consequence of this zero crossing can be seen in figure 5. From the right panel of this figure one can see clearly that while for zero flat dimensions the dominant contribution to the per-channel contribution of the energy comes from the low-k region, for one flat dimension the dominant contribution comes from the region $k \cdot w > M$, where w is the characteristic width of the potential. This result explains why the interchange of summation and integration fails on the real axis for one flat dimension (and not for zero); if one integrates first over momentum, one obviously is able to catch the dominant large-momentum contribution. If one sums over channels first, however, one finds that for $M > k \cdot w$ the phase shifts drop off exponentially. Hence it seems sufficient to sum up to $M_{\max} \approx (2 \dots 3)k \cdot w$. Since the momentum integral will be terminated at some finite value $k_{\max}$, this procedure obviously misses important contributions from channels with $M > M_{\max}$. Hence, integration and summation cannot be interchanged[3]. Another piece of evidence is that the momentum integrand obtained from summing over channels first is—as far as we can tell from our numerical investigations—strictly positive for any prescribed finite momentum, which is in direct contradiction to the sum rule from (9).

4. Conclusions

We have shown that for finite quantities one can successfully replace Born approximations in one theory by Born approximations from another as long as the Feynman diagrams show the

[3] During our investigation of the second-order Born approximation phase shifts, we noted that $k^3\delta_M^{(2)}(k)$ is not a function of k and M individually, but is just a function of k/M (plus corrections, but those are very small for $M > 1$). This allows to map $\sum_M \delta_M^{(2)}(k)$ via the (leading order) Euler–Maclarin formula to $\int \mathrm{d}k\, k\delta_M^{(2)}(k)$ which might ultimately explain why on the real axis the sum over channels is so small. For definite conclusions though it will be necessary to investigate the next-to-leading orders.

same divergences. This is a major improvement over existing formalisms: one does not have to deal with problems that originate from non-vanishing structures at spatial infinity, but still one knows exactly what to add in to implement the renormalization conditions known from perturbation theory. In these conference proceedings, we have focused on a pathological case where the ultimate object of interest does not strictly speaking exist (as it is UV divergent); but the apparent convergence of the vacuum polarization diagram in the phase shift formulation was important to understand. This (erroneous) convergence only occurs when formulating the vacuum polarization energy in terms of scattering data for real momenta. Analytically continuing to the imaginary axis yields the UV structure that is consistent with the analysis of the Feynman diagrams without any further subtleties like ordering of limits. Though this favours the formulation in terms of imaginary momenta it contains drawbacks that we did not go into—one needs to sum a lot more channels and it *may* pose challenges for fermions.

Acknowledgments

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